Burnside's Formula

Theorem 0.1 (Burnside's Formula)**.** *Let G be a finite group and A a finite G-set. If r is the number of orbits in A under G, then*

$$r = \frac{1}{|G|} \sum_{g \in G} |A_g|,$$

where $A_g := \{a \in A : \ g \cdot a = a\}$.

1. Counting via group actions

Question. Given a frame of shape of a regular k-gon and n colors, how many ways are there with which we can paint the frame?

Solution via group action. Assume first that the frame is fixed and not allowed to rotate or flip. Let A be the set of all possible colorings of the frame.

(1) Suppose that we consider two colorings as the same if we can obtain one coloring from the other coloring by rotation or flip. In the group-theoretic language, this means that two colorings are the same if after we apply the action of the symmetry group D_{2k} on one coloring, the result is the other coloring.
(2) In mathemtical terms, the above discussion says that the group D_k acts on A and we regard a_1 and a_2 in A as the same if $a_1 = \sigma \cdot a_2$ for some $\sigma \in D_{2k}$.

In summary, let A denote the set of all possible colorings of a fixed frame of shape of a regular k-gon. Let D_{2k} be the symmetry group of a regular k-gon. Then D_{2k} acts on A.

We say two elements (colorings) a_1 and a_2 in A are equivalent if $a_1 = g \cdot a_2$ for some $g \in D_{2k}$. Thus, the number of ways to paint the frame with rotation and refetion allowed is the same as the number of orbits.

By Burnside's formula, the number then is equal to

$$\frac{1}{|D_{2k}|} \sum_{g \in D_k} |A_g|,$$

where A_g consists of colorings that remains the same after we apply the action of g on them.

Example 1.1. *Find the number of distinguishable ways the edges of a square can be painted if n different colors of paint are available and the same color can be used on any number of edges.*

2. A general approach to coloring problems

To count the number of possible ways to paint an object with n colors, we may proceed as follows.

(1) Label all the faces/edges that are to be painted by numbers $1, \dots, k$.
(2) Determine the symmetry group G and represent the symmetries as permutations of k letters.
(3) Given a permutation $\sigma \in G$, denote by $f(\sigma)$ the number of orbits under σ. (For example, $f(1) = k$.)

(4) Then by Burnside's formula, the number of possible colorings is

$$\frac{1}{|G|}\sum_{\sigma\in G} n^{f(\sigma)}.$$

3. Exercise

(1) Find a general formula for the number of ways to paint a regular k-gon with n colors. (The formula will involve the divisors of k.)

(2) Find the number of distinguishable ways the edges of a square can be painted if 6 different colors of paint are available and

   (a) the same color can be used on any number of edges.

   (b) no color is used more than once.

(3) Find a general formula for the number of ways to paint a regular cube.