Elliptic Functions and Complex Tori

[Koblitz]: I.4-6

Goals

- For a given lattice L in $\mathbb{C}$, the complex torus C/L is an elliptic curve.
  - Let $\mathcal{E}(L)$ be the field of elliptic functions for L. Then

$$\mathcal{E}(L) = \mathbb{C}(\wp, \wp').$$

  - The associated Weierstrass $\wp$-function and its derivative $\wp'$ give a bijection from $\mathbb{C}/L$ to the elliptic curve

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2(L)\wp(z) - g_3(L),$$

    for some computable constants $g_2(L)$ and $g_3(L)$.
- Every complex elliptic curve can be identified with a complex torus.
  - For a given elliptic curve with the equation

$$y^2 = 4x^3 - ax - b, \quad a^3 - 27b^2 \neq 0,$$

    there exists a lattice L in $\mathbb{C}$ such that $a = g_2(L)$ and $b = g_3(L)$.

Elliptic Functions

Definitions. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.

- The parallelogram

$$\Pi_L := \{a\omega_1 + b\omega_2 : 0 \leq a \leq 1, 0 \leq b \leq 1\}$$

  is called a fundamental parallelogram for ω_1 and ω_2.
- An elliptic function with respect to L if
  - $f(z)$ is meromorphic on $\mathbb{C}$,
  - $f(z + \omega) = f(z)$ for all $\omega \in L$.

  We call any element in L a *period*.

Facts. The set $\mathcal{E}(L)$ of all elliptic functions for a lattice L is a field and is closed under differentiation.

Some Properties

Proposition I.4.3. If an elliptic function f has no pole in some fundamental parallelogram, f is a constant.

Proposition I.4.4. For a given lattice L, let $f \in \mathcal{E}(L)$. If f has no poles on the boundary of $P := \alpha + \Pi_L$ for some $\alpha \in \mathbb{C}$, the sum of the residues of f in P is zero.

Collary.

- **Theorem.** Every non-constant elliptic function has at least two poles in a fundamental parallelogram, counting multiplicities.
- **Proposition I.4.5.** For a given lattice L, let $f \in \mathcal{E}(L)$. If f has no poles or zeros on the boundary of $P := \alpha + \Pi_L$ for some $\alpha \in \mathbb{C}$, then the number of zeros of f in P is equal to the number of poles, each counted with multiplicity.

Proofs

Proposition I.4.3. If an elliptic function f has no pole in some fundamental parallelogram, f is a constant.

Ideas of the proof. By periodicity and Liouville's theorem.

Liouville's Theorem. A bounded entire function is constant.

Proofs

Proposition I.4.4. For a given lattice L, let $f \in \mathcal{E}(L)$. If f has no poles on the boundary of $P := \alpha + \Pi_L$ for some $\alpha \in \mathbb{C}$, the sum of the residues of f in P is zero.

Ideas of the proof. By Cauchy's residue theorem:

$$\sum_{w \in P} \mathrm{res}_f(w) = \frac{1}{2\pi i} \oint_{\partial P} f(z)dz = 0.$$

Proofs

Collary.

- Theorem. Every non-constant elliptic function has at least two poles in a fundamental parallelogram, counting multiplitities.
- Proposition I.4.5. For a given lattice L, let $f \in \mathcal{E}(L)$. If f has no poles or zeros on the boundary of $P := \alpha + \Pi_L$ for some $\alpha \in \mathbb{C}$, then the number of zeros of f in P is equal to the number of poles, each counted with multiplicity.

Ideas of the proofs. By Proposition I.4.4 and the property....

- the residue of a meromorphic function f at an isolated singularity a is the coefficient a_{-1} of the Laurent series of f at a.
- $\sum_{w \in P} \mathrm{res}_{\frac{f'}{f}}(w) = \#\{\text{zeros in } P\} - \#\{\text{poles in } P\}$.

Constrution of Elliptic Functions

Lemmas. Let L be a lattice in $\mathbb{C}$.

- If s is real, the series

$$\sum_{\omega \in L - \{0\}} \frac{1}{\omega^s}$$

  converges absolutely if, and only if, $s > 2$.

- If $s > 2$ and $R > 0$, the series

$$\sum_{|\omega| > R} \frac{1}{(z - \omega)^s}$$

  converges absolutely and uniformly in the disk $|z| \leq R$.

Theorem. Let f be defined by the series

$$f(z) = \sum_{\omega \in L} \frac{1}{(z - \omega)^3}.$$

Then $f \in \mathcal{E}(L)$ and f has a pole of order 3 at each period $\omega \in L$.

Weierstrass $\wp$-function

Definition. Let L be a lattice in $\mathbb{C}$. The Weierstrass $\wp$-function is defined by the series

$$\wp(z) = \wp(z; L) := \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$

Proposition I.4.6. The double series

$$\sum_{\omega \in L - \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

converges absolutely and uniformly for z in any compact subset of $\mathbb{C} - L$.

Proposition I.4.7. $\wp(z) \in \mathcal{E}(L)$, and its only pole is a double pole at each period $\omega \in L$. Moreover, $\wp(z)$ is an even function.

The Field of Elliptic Functions

Proposition I.5.9. Let $f \in \mathcal{E}(L)$. If f is an even function, then $f \in \mathbb{C}(\wp)$.

Ideas.

- if f has a zero (or pole) of order m at z_0, then it has a zero (or pole) of order m at $-z_0$.
- if $z_0 \equiv -z_0 \mod L$, then the order m must be even.
- Choose a set of representatives mod L for the zeros and poles of f not in L to be $\{z_1, \ldots, z_k, -z_1, \ldots, -z_k, z_{k+1}, \ldots, z_n\}$ such that

$$z_i \not\equiv -z_i,\ 1 \le i \le k, \quad z_i \equiv -z_i \not\equiv 0, k < i \le n.$$

- Let m_i be the order of f at z_i. The function

$$g(z) := \prod_{i=1}^{k} (\wp(z) - \wp(z_i))^{m_i} \prod_{i=k+1}^{n} (\wp(z) - \wp(z_i))^{m_i/2}$$

  and $f(z)$ have exactly the same zeros and poles.

Proposition I.5.8. $\mathcal{E}(L) = \mathbb{C}(\wp, \wp')$.

The Laurent Expansion of $\wp$ near 0

Proposition. Let $r = \min\{|\omega| : \omega \in L - \{0\}\}$. Then for z such that $0 < |z| < r$ we have

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2n+2}(L)z^{2n},$$

where $G_n(L) := \sum_{\omega \neq 0} \frac{1}{\omega^n}$, for $n \geq 3$.

Idea.

$$\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] = \frac{1}{\omega^2}\left[\frac{1}{(1-z/\omega)^2} - 1\right]$$

Differential Equation Satisfied by $\wp$

Proposition. The functions $\wp$ and $\wp'$ satisfy the relation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2(L)\wp(z) - g_3(L),$$

where $g_2(L) = 60G_4(L)$ and $g_3(L) = 140G_6(L)$.

Idea. Comparing the expansions

$$\begin{aligned}\wp(z) =& z^{-2} + 3G_4(L)z^2 + 5G_6(L)z^4 + \cdots \\ \wp'(z) =& -2z^{-3} + 6G_4(L)z + 20G_6(L)z^3 + \cdots ,\end{aligned}$$

we obtain that

$$(\wp'(z))^2 - 4\wp(z)^3 + 60G_4(L)\wp(z) = -140G_6(L) + 108G_4(L)^2z^2 + \cdots$$

Complex Tori as Elliptic Curves ?

Proposition. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and set $\omega_3 = \omega_1 + \omega_2$. Then the cubic equation satisfied by $\wp$ and $\wp'$, is

$$\begin{aligned} y^2 &= 4x^3 - g_2(L)x - g_3(L) \\ &= 4(x - e_1)(x - e_2)(x - e_3), \quad e_i = \wp(\omega_i/2). \end{aligned}$$

This equation is nonsingular.

Proposition I.6.10. For a given lattice L in $\mathbb{C}$, there is a (an analytic) bijection between the complex trous $\mathbb{C}/L$ and the elliptic curve $y^2 = 4x^3 - g_2(L)x - g_3(L)$ in $\mathbb{P}^2(\mathbb{C})$.