Complex Tori, Elliptic Curves as Riemann Surfaces

Riemann Surface

A second-countable Hausdorff topological space together with a complex structure is a Riemann surface.

Let M be a one-dimensional connected complex manifold. A coordinate chart (U, z) is an open subset U of M, together with a homeomorphism $z : U \mapsto \mathbb{C}$ onto an open subset of $\mathbb{C}$.

A Riemann surface is a one-dimensional connected complex manifold M with a set of coordinate charts $\{(U_\alpha, z_\alpha)\}_{\alpha \in I}$, where I is an index set, such that

- $\{U_\alpha\}_{\alpha \in I}$ forms an open cover of M,
- the transition functions

$$w_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \mapsto z_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic whenever $U_\alpha \cap U_\beta \neq \phi$.

Examples

The followings are Riemann surfaces.

- The complex plane $\mathbb{C}$, the upper half plane $\mathbb{H}$, the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
- The complex projective line $\mathbb{P}^1(\mathbb{C})$. We can choose the local charts to be $(U_1, f_1), (U_2, f_2)$, where

$$U_1 = \{[x, y] : x \neq 0\}, \quad f_1([x : y]) = y/x,$$

$$U_2 = \{[x : y] : y \neq 0\}, \quad f_2([x : y]) = x/y.$$

- The complex torus. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. Let $A = \{a\omega_1 + b\omega_2 : 0 \leq a, b < 1\}$ be a complete set of coset representatives of $\mathbb{C}/L$. Let $r = \min\{|\omega_1|/2, |\omega_2|/2\}$. For $\alpha \in A$ set $U_\alpha = \{z + L : |z - \alpha| < r\}$. Then $\mathbb{C}/L$ is a Riemann surface with charts $\{U_\alpha, z_\alpha\}_{\alpha \in A}$.

- Let X and Y be two Riemann surfaces. A continuous mapping $f : X \mapsto Y$ is called holomorphic if for every local coordinate (U, z) on X and every local coordinate (V, ζ) on Y with $U \cap f^{-1}(V) \neq \phi$, the mapping

$$\zeta \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \mapsto \zeta(V)$$

  is holomorphic.
- An isomorphism of Riemann surfaces is a bijective holomorphic map whose inverse is also holomorphic.

Proposition I.6.10. For a given lattice L in $\mathbb{C}$, we let E_L to be the elliptic curve

$$Y^2 Z = 4X^3 - g_2(L)XZ^2 - g_3(L)Z^3.$$

The map

$$z + L \longmapsto \begin{cases} [\wp(z), \wp'(z), 1], & \text{if } z \neq 0, \\ [0, 1, 0], & \text{if } z = 0. \end{cases}$$

is an isomorphism of Riemann surfaces $\mathbb{C}/L$ and $E_L(\mathbb{C})$.

Theorem. Every elliptic curve E over $\mathbb{C}$ is isomorphic to E_L for some lattice L.

Holomorphic Maps Bewteen Complex Tori

Let L and L' be two lattices in $\mathbb{C}$.

Proposition. Let $\phi : \mathbb{C}/L \longrightarrow \mathbb{C}/L'$ be a holomorphic map. Then there exist $a, b \in \mathbb{C}$ with $aL \subseteq L'$ such that

$$\phi(z + L) = az + b + L'.$$

The map is invertibel if and only if $aL = L'$.

Corollary. Let $\phi : \mathbb{C}/L \longrightarrow \mathbb{C}/L'$ be a holomorphic map, which is given by $\phi(z + L) = az + b + L'$, $aL \subseteq L'$. Then the following are equivalent:

- ϕ is a group homomorphism.
- $\phi(0) = 0$
- $\phi(z + L) = az + L'$.

Isogeny. A nonzero holomorphic homomorphism bewteen complex tori is called an isogeny.

Corollary. The Riemann surfaces $\mathbb{C}/L$ and $\mathbb{C}/L'$ are isomorphic if and only if $aL = L'$ for some $a \in \mathbb{C}^{\times}$.

Notations/Definitions.

- For $n \geq 3$, define $G_n(L) := \sum_{\omega \neq 0} \frac{1}{\omega^n}$, and $G_n(\tau) := G_n(\mathbb{Z}\tau + \mathbb{Z})$ with $\tau \in \mathbb{H}$. The functions $G_n(L)$ and $G_n(\tau)$ are called Eisenstein series.
- The function $\Delta(L) := g_2(L)^3 - 27g_3(L)^2$ is called the discriminant of the elliptic curve $E_L : \; y^2 = 4x^3 - g_2(L)x - g_3(L)$.
- The function $j(L) := 1728g_2(L)^3/\Delta(L)$ is called the elliptic j-invariant.

Theorem. Every elliptic curve E over $\mathbb{C}$ is isomorphic to E_L for some lattice L.

It is known that every compact Riemann surface X can be decomposed into triangles such that two triangles either do not intersect or share a common vertex or a common edge.

Definitions. Denote V the number of vertices, E the number of edges, and F the number of faces.

- The number
$$\chi(X) := V - E + F$$
is an invariant and is called the Euler characteristic of X.
- Let g be the integer satisfying
$$\chi = 2 - 2g.$$
This number g is called the genus of X, and can be interpreted as the number of “holes” of X.

Example

- The genus of $\mathbb{P}^1(\mathbb{C})$ is 0.
- The genus of a complex torus is 1.

Holomorphic and meromorphic functions

Let X and Y be two Riemann surfaces. Definitions.

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$$\zeta \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \mapsto \zeta(V)$$

  is holomorphic.
- A holomorphic mapping into $\mathbb{C}$ is called a holomorphic function. A holomorphic mapping into $\mathbb{P}^1(\mathbb{C})$ is called a meromorphic function.
- A function $f : U \longrightarrow \mathbb{P}^1(\mathbb{C})$ is called meromorphic if it is holomorphic at every point where it has a finite value, whereas, near every point z_0 with $f(z_0) = \infty$, $f(z) = \phi(z)/(z - z_0)^n$ for some holomorphic function ϕ, defined and non-zero around z_0. The positive number n is the order of the pole at z_0.