Suggested Problems and Homework due on 11/15

(1) Examples in Section 8.1 and Exercises: 8, 10.
(2) Examples in Section 8.2 and Exercises: 1, 2, 5.
(3) Examples in Section 8.3, Factorization in Gaussian Integers, and Exercises: 5, 6.
(4) Examples in Section 9.2 and Exercises: 9, 10.
(5) Suppsoe that D is an integral domain.
  (a) Show that if D is a field, then the polynomial ring $D[x]$ is a PID.
  (b) Show that if $D[x]$ is a PID, then D is a field.
(6) (a) Let D be a PID and I, J nonzero ideals of D. Show that $IJ = I \cap J$ if and only if $I + J = D$.
  (b) Show that $\mathbb{Z}/900\mathbb{Z}$ is isomorphic to $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/100\mathbb{Z}$ as rings.
(7) (a) Show that the ring $\mathbb{Q}[X]/\langle x^2+2\rangle$ is isomorphic to $\mathbb{Q}[\sqrt{-2}] := \{a+b\sqrt{-2} : a, b \in \mathbb{Q}\}$.
  (b) Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.
  (c) Prove that $\mathbb{Z}[\sqrt{-6}]$ is not a UFD.
(8) (a) Is the ideal $\langle 3+i\rangle$ a maximal of $\mathbb{Z}[i]$? Describe the quotient ring $\mathbb{Z}/\langle 1+3i\rangle$.
  (b) Use the Euclidean algorithm in $\mathbb{Z}[i]$ to find a GCD of $9+13i$ and $19+13i$.
  (c) Prove that 3 is irreducible in $\mathbb{Z}[i]$. Find also the number of elements and the characteristic of the field $\mathbb{Z}[i]/\langle 3\rangle$.
(9) Let $f(x) = x^4 + 4x^3 + 3x + 1$ and $g(x) = 2x^2 - x + 2$ in $\mathbb{Z}_7[x]$.
  (a) Find $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg r(x) < \deg g(x)$ for $f(x)$ and $g(x)$
  (b) Determine the GCD of $f(x)$ and $g(x)$ in $\mathbb{Z}_7[x]$ and write it as a "linear combination" of $f(x)$ and $g(x)$ if the GCD exists.
(10) Let $\omega = e^{2\pi i/6} = \frac{1+\sqrt{-3}}{2}$.
  (a) Show that the ring $\mathbb{Q}[X]/\langle x^2 - x + 1\rangle$ is isomorphic to $\mathbb{Q}[\omega] := \{a + b\omega : a, b \in \mathbb{Q}\}$.
  (b) Show that $\mathbb{Z}[\omega]$ is a Euclidean domain with respect to the norm $N(a) = a\bar{a}$.