ALGEBRA I-MATH 7210

MIDTERM EXAM 2: DUE ON 11/20 12:00 PM

Note.

- Please provide details.
- You can use any mathematical software packages or computational systems to verify your calculations.
- Do all the problems of Part I. You can use your textbook, handouts, and notes. Do not consult any other sources.
- Do one problem of Part II. You can use your textbook, handouts, notes, and discsuss with your classmates. No other resources are permitted. The solutions you hand in must be written in your own words.
- Upper Bound: 100 points.

1. Part I

(1) (15 points) How many solutions of $x^{34} = 1$ in $(\mathbb{Z}/7210\mathbb{Z})^{\times}$ are there? Explain.

(2) (15 points) Let G be a group of order 12. Assume that G does not contain a subgroup of order 6. Prove that G has only one Sylow 2-subgroup in G and that its Sylow 2-subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(3) (20 points) Let $\mathbb{F}$ be the field $\mathbb{Z}/2\mathbb{Z}$.
   (a) Find all the irreducible polynomials of degree 3 in $\mathbb{F}[x]$.
   (b) Let $f(x) = x^3 + x^2 + 1$. Find the inverse of $x + 1 + \langle f(x) \rangle$ in $\mathbb{F}[x]/\langle f(x) \rangle$, expressed as $p(x) + \langle f(x) \rangle$ for some polynomial $p(x)$.
   (c) Find a generator for the multiplicative group $(\mathbb{F}[x]/\langle f(x) \rangle)^{\times}$.

(4) (15 points) Classify the groups of order $5 \cdot 7 \cdot 37$.

(5) (15 points) For a positive integer n, let

$$X_n = \{(x_1, x_2, x_3, x_4, x_5) : \ x_j \in \mathbb{Z}, \ 1 \leq x_j \leq n\}.$$

Let the alternating group A_5 act on X_n by

$$\sigma \cdot (x_1, \ldots, x_5) = (x_{\sigma(1)}, \ldots, x_{\sigma(5)}).$$

Find the number of orbits.

2. Part II

(1) (25 points)

   (a) Prove that any subgroup of $\mathbb{Z} \times \mathbb{Z}$ of finite index is also isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

   (b) Let H_1 and H_2 be two subgroups of $\mathbb{Z} \times \mathbb{Z}$ of finite index. From Part (a) we know that H_j, $j = 1, 2$, is generated by (a_j, b_j) and (c_j, d_j) for some $a_j, b_j, c_j, d_j \in \mathbb{Z}$. Prove that $a_1 d_1 - b_1 c_1 \neq 0$ and that $H_1 = H_2$ if and only if there exist integers $a, b, c, d \in \mathbb{Z}$ with $ad - bd \in \{\pm 1\}$ such that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

   (c) Assume that H is a subgroup of $\mathbb{Z} \times \mathbb{Z}$ of finite index. Prove that there exist positive integers a, d and an integer b with $0 \leq b < d$ such that H is generated by (a, b) and $(0, d)$.

   (d) Let p be a prime. Prove that the number of distinct subgroups of index p in $\mathbb{Z} \times \mathbb{Z}$ is $p + 1$.

(2) (25 points) Let $\omega = e^{2\pi i/6} = \frac{1+\sqrt{-3}}{2}$.

   (a) Show that $\mathbb{Z}[\omega]$ is a Euclidean domain with respect to the norm $N(a) = a\bar{a}$.

   (b) Prove that a prime $p > 3$ in $\mathbb{Z}$ can be written as $p = a^2 + ab + b^2$ for some $a, b \in \mathbb{Z}$ if and only if $p \equiv 1 \mod 3$.