Modular Curves as Riemann Surfaces

[Koblitz]: III.1 – Part 2

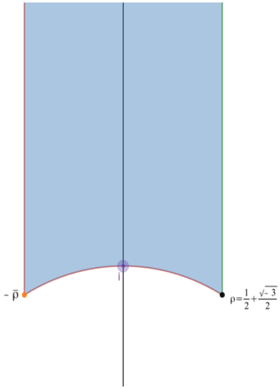
Goal.

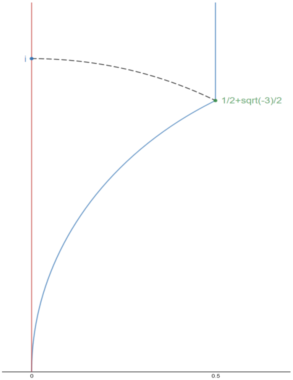
- The quotient space $X_0(1) := \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^*$ is a compact Riemann surface.
- The genus of $X_0(1)$ is 0.
- The field of meromorphic functions on $X_0(1)$ is $\mathbb{C}(j)$, where j is the elliptic j-invariant. (Will see in Proposition III.2.12.)

Recall. A second-countable Hausdorff topological space together with a complex structure is a Riemann surface.

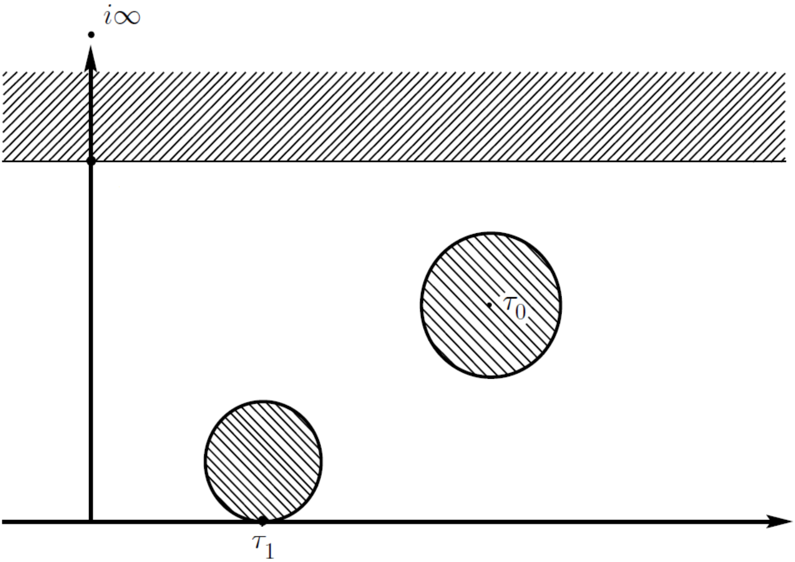
Recall. every compact Riemann surface X can be decomposed into triangles such that two triangles either do not intersect or share a common vertex or a common edge. Denote V the number of vertices, E the number of edges, and F the number of faces. The genus of X, $g(X)$, is the number g satisfying

$$V - E + F = 2 - 2g.$$

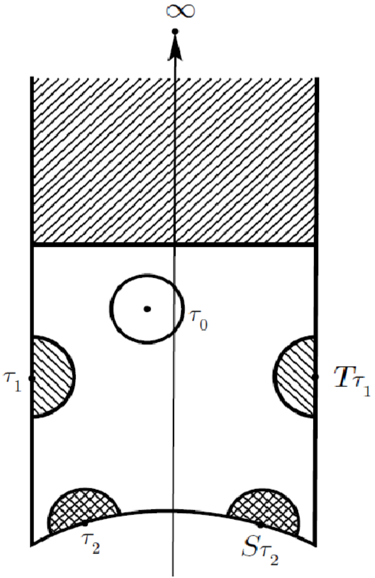




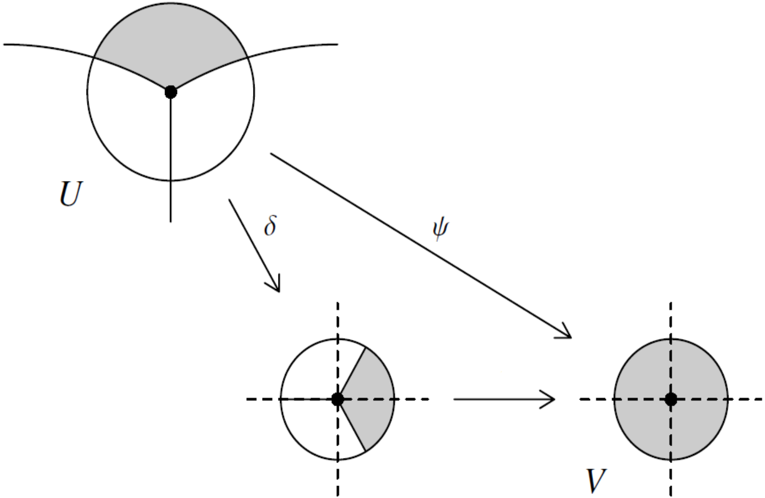
Fundamental system of neighborhoods in $\mathbb{H}$. (pictures from Cohen-Strömberg's book "Modular Forms")



Fundamental system of neighborhoods in F.



Neighborhood around the elliptic point ρ (order-3) and its local parameter. (picture from Diamond-Shurman's book "A First Course in Modular Forms")



The Complex Structure of $X_0(1)$

The following maps give to $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^*$ the structure of a Riemann Surface.

point τ_0	local parameter
$\tau_0 \neq i, \rho, \infty$	$\tau \mapsto \frac{\tau - \tau_0}{\tau - \overline{\tau_0}}$
i	$\tau \mapsto \left(\frac{\tau - i}{\tau - \overline{i}}\right)^2$
ρ	$\tau \mapsto \left(\frac{\tau - \rho}{\tau - \overline{\rho}}\right)^3$
∞	$\tau \mapsto q = e^{2\pi i \tau}$