# Modular Forms – Part 1

[Koblitz]: III.2

# Slash Operator

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. For an integer $k$ and a meromorphic function $f : \mathbb{H} \mapsto \mathbb{C}$ we let the notation $f \mid [\gamma]_k$ denote the slash operator

$$f \mid [\gamma]_k = (\det \gamma)^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

The factor $c\tau + d$ is called the automorphy factor.
(If the weight $k$ is clear from the context, we often write simply $f \mid \gamma$.)

Lemma. For $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{R})$ and a meromorphic function $f : \mathbb{H} \mapsto \mathbb{C}$, we have

$$f \mid [\gamma_1\gamma_2]_k = \left(f \mid [\gamma_1]_k\right) \mid [\gamma_2]_k.$$

# Modular Forms

**Definition.** Let $G$ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of a finite index, and $k$ be any integer. A holomorphic function $f : \mathbb{H} \mapsto \mathbb{C}$ is a modular form of weight $k$ with respect to $G$ if

(1) $f(\tau) \bigm| [\gamma] = f(\tau)$ for all $\tau \in \mathbb{H}$ and $\gamma \in G$,
(2) $f(\tau)$ is holomorphic at every cusp.

The set of all modular forms of weight $k$ with respect to $G$ is denoted by $M_k(G)$. If, in addition to (1) and (2), the function also satisfies

(3) $f$ vanishes at every cusp,

then the function $f$ is a cusp form of weight $k$ with respect to $G$. The set of all cusp forms of weight $k$ on $G$ is denoted by $S_k(G)$.

**Proposition.** The sets $M_k(\Gamma)$ and $S_k(\Gamma)$ are vector spaces over $\mathbb{C}$.

# Examples

For even integers $k \geq 4$, the Eisenstein series

$$G_k(\tau) := \sum_{\substack{m,n\in\mathbb{Z},\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}$$

satisfy

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau)$$

for all $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, and therefore are modular forms of weight $k$ with respect to $\mathrm{PSL}_2(\mathbb{Z})$.
Write $g_2 = 60G_4$ and $g_3 = 140G_6$. Then

- $g_2^3$ and $g_3^2$ are both modular forms of weight 12;
- $\Delta = g_2^3 - 27g_3^2$ is a non-zero modular form that vanishes at the unique inequivalent cusp $\infty$ for $PSL_2(\mathbb{Z})$ and thus $\Delta$ is a cusp form of weight 12.

# Remark (Take $G$ that contains $\pm I$ for example)

Let $a/c \in \mathbb{P}^1(\mathbb{Q})$ be a cusp and choose $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$.

Then a function $f$ satisfies condition (1) if and only if the function $g(\tau) = f \big|_k [\sigma]$ is invariant under the action of $\sigma^{-1}\Gamma\sigma$ since

$$\left(f\big|[\sigma]\right)\big|[\sigma^{-1}\gamma\sigma] = \left(f\big|[\gamma]\right)\big|[\sigma] = f\big|[\sigma]$$

for all $\gamma \in G$. In particular, $g(\tau)$ is invariant under the substitution $\tau \mapsto \tau + h$, where $h$ is the smallest positive integer such that $\sigma T^h \sigma^{-1} \in \overline{G}$ (called width or ramification index of the cusp $a/c$). Let

$$\sum_{n\in\mathbb{Z}} a_n q_h^n, \quad q_h = e^{2\pi i\tau/h}$$

be the Fourier expansion of $g(\tau)$. Then we say $f$ is holomorphic at $a/c$ provided that $a_n = 0$ for all $n < 0$. Moreover, with this setting, condition (3) means that $a_n = 0$ for all $n \leq 0$ for each cusp $a/c$.

# Modular Functions

Definition. Let $G$ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of a finite index. A meromorphic function $f : \mathbb{H}^* \mapsto \mathbb{C}$ is a modular function if

$$f(\gamma\tau) = f(\tau), \quad \text{for all } \tau \in \mathbb{H}, \gamma \in G.$$

Example. The $j$-invariant $j := 1728 g_2^3/\Delta$ is a modular function on $\mathrm{PSL}_2(\mathbb{Z})$.

# Modular Forms on $\mathrm{PSL}_2(\mathbb{Z})$

Goal. Denote $\overline{\Gamma}$ the group $\mathrm{PSL}_2(\mathbb{Z})\}$. In the following, we will see and/or show

1. Fourier expansions of Eisenstein series;
2. $M_k(\overline{\Gamma}) = S_k(\overline{\Gamma}) \oplus \mathbb{C}G_k$ and dimensions of $M_k(\overline{\Gamma})$ ;
3. Proposition III.2.10: Modualr forms on $\overline{\Gamma}$ can be expressed in terms of Eisenstein series $G_4$ and $G_6$.
4. Proposition III.2.11: The elliptic $j$-funEisenstein seriesction gives a bijection from $X_0(1)$ to $\mathbb{P}^1(\mathbb{C})$.
5. Proposition III.2.12: The field of meromorphic functions on $X_0(1)$ is $\mathbb{C}(j)$.

# Normalized Eisenstein series

We define the (normalized) Eisenstein series of weight $k$ by
$E_k(\tau) := G_k(\tau)/2\zeta(k)$.

$$G_k(\tau) := \sum_{\substack{c,d\in\mathbb{Z},\\(c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k} = \sum_{n>0}\sum_{\gcd(c,d)=n} \frac{1}{(c\tau+d)^k}$$

$$= \sum_{n\in\mathbb{N}} \frac{1}{n^k} \sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k} = \zeta(k) \sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}$$

$$= 2\zeta(k)\left(1 + \sum_{\substack{\gcd(c,d)=1\\ c>0}} \frac{1}{(c\tau+d)^k}\right),$$

where $\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s}$ is the Riemann zeta function.

Thus,

$$E_k(\tau) := \frac{G_k(\tau)}{2\zeta(k)} = \frac{1}{2} \sum_{\substack{\gcd(c,d)=1 \\ c,d\in\mathbb{Z}}} \frac{1}{(c\tau + d)^k}$$

Remark: $E_2$. We can define Eisenstein sereis $G_2$ and $E_2$ in a similar way, but they are not modular forms:

$$E_2(\gamma\tau) = (c\tau+d)^2 E_2(\tau) - \frac{6i}{\pi}\cdot c(c\tau+d), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}).$$

To obtain the Fourier expansions of Eisenstein series, we will use the Lipschitz summation formula, which can be derived from Poisson summation formula.

## Theorem

*Let $k \geq 2$ be an even integer. Let $E_k(\tau)$ be the normalized Eisenstein series of weight $k$. Then we have*

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\Gamma(k)\zeta(k)} \sum_{r=1}^{\infty} \frac{r^{k-1} q^r}{1 - q^r} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

*where $q = e^{2\pi i \tau}$ and $B_k$ are the Bernoulli numbers.*

**Corollary.** For positive integers $n$,

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m)$$

and

$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m).$$

**Poisson Summation Formula.** Suppose that $f(t)$ is continuous and of bounded variation such that

$$\int_{\mathbb{R}} |f(t)|\, dt < \infty.$$

Then one has

$$\sum_{n\in\mathbb{Z}} f(t+n) = \lim_{N\to\infty} \sum_{|n|\le N} \hat{f}(n) e^{2\pi i n t}$$

for all $t \in [0, 1)$, where

$$\hat{f}(n) = \int_{\mathbb{R}} f(t) e^{-2\pi i n t}\, dt.$$

**Lipschitz summation formula.** Let $k \in \mathbb{Z}_{>1}$. For $\tau \in \mathbb{H}$, we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{\Gamma(k)} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r\tau}.$$

**Sketch of Proof.** By the Poisson summation formula we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \sum_{r\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x+\tau)^k}\, dx.$$

- When $r = 0$, the integral is 0.
- When $r < 0$, we move the line of integration to the horizontal line $\operatorname{Im} x = T$ and let $T$ tend to infinity. The integral is 0.
- When $r > 0$, we move the line of integration to the line $\operatorname{Im} x = -T$ and let $T \to \infty$. By doing so, we cross the pole of the integrand at $x = -\tau$. Thus, the integral is

$$-2\pi i \cdot \operatorname{Res}_{x=-\tau} \left( \frac{e^{-2\pi i r x}}{(x+\tau)^k} \right) = \frac{(-2\pi i)^k}{(k-1)!} r^{k-1} e^{2\pi i r\tau}.$$

## Proof of Theorem (use $G_k$).

$$G_k(\tau) := \sum_{\substack{c,d\in\mathbb{Z},\\(c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k} = \sum_{d\neq 0}\frac{1}{d^k} + \sum_{\substack{c\neq 0\\ d\in\mathbb{Z}}}\frac{1}{(c\tau+d)^k}$$

$$= 2\zeta(k) + 2\sum_{\substack{c>0\\ d\in\mathbb{Z}}}\frac{1}{(c\tau+d)^k}$$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{\Gamma(k)}\sum_{r>0} r^{k-1}\sum_{c>0} q^{rc}$$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{\Gamma(k)}\sum_{r=1}^{\infty}\frac{r^{k-1}q^r}{1-q^r}.$$

Thus,

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\Gamma(k)\zeta(k)}\sum_{r=1}^{\infty}\frac{r^{k-1}q^r}{1-q^r}.$$

## Proof of Theorem.

Due to the exclusion-inclusion principle, we can rewrite $E_k(\tau)$ as

$$E_k(\tau) = 1 + \sum_{c>0} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(c\tau + d)^k}$$

$$= 1 + \sum_{c>0} \sum_{m|c} \mu(m) \sum_{d \in \mathbb{Z}, m|d} \frac{1}{(c\tau + d)^k}$$

$$= 1 + \sum_{c>0} \sum_{m|c} \frac{\mu(m)}{m^k} \sum_{d \in \mathbb{Z}} \frac{1}{(c\tau/m + d)^k},$$

where $\mu(m)$ is the Möbius function.

Applying Lipschitz summation formula, we obtain

$$E_k(\tau) = 1 + \frac{(-2\pi i)^k}{(k-1)!} \sum_{c>0} \sum_{m|c} \frac{\mu(m)}{m^k} \sum_{r \in \mathbb{N}} r^{k-1} e^{2\pi i r c\tau/m}$$

Thus

$$E_k(\tau) = 1 + \frac{(-2\pi i)^k}{(k-1)!} \sum_{c\in\mathbb{N}} \sum_{m|c} \frac{\mu(m)}{m^k} \sum_{r\in\mathbb{N}} r^{k-1} e^{2\pi i r c\tau/m}$$

$$= 1 + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^k} \sum_{r=1}^{\infty} \sum_{c=1}^{\infty} r^{k-1} e^{2\pi i r c\tau}.$$

Noticing

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m^k} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^k}\right) = \frac{1}{\zeta(k)},$$

we see that

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\Gamma(k)\zeta(k)} \sum_{r=1}^{\infty} \frac{r^{k-1} q^r}{1 - q^r},$$

where we set $q = e^{2\pi i\tau}$.

The series can also be written as

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\Gamma(k)\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n)$ is the sum of divisors function defined by

$$\sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}.$$

For $n = 2m \in 2\mathbb{Z}^+$, the zeta values are given by

$$\zeta(2m) = \frac{(-1)^{m-1}}{(2m)!} 2^{2m-1} \pi^{2m} B_{2m},$$

where $B_n$ is the Bernoulli number and can be determined by the Tayler expansion

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

**Proposition III.2.8.** Let $f(z)$ be a meromorphic modular form of weight $k$ for $\mathrm{PSL}_2(\mathbb{Z})$. Then

$$v_\infty(f) + \sum_{P \in \mathbb{H}} \frac{1}{e_P} v_P(f) = \frac{k}{12},$$

where

$$v_P(f) = \mathrm{ord}_p(f) := \begin{cases} m, & \text{if } f \text{ has a zero of order } m \text{ at } P, \\ -m, & \text{if } f \text{ has a pole of order } m \text{ at } P, \\ 0, & \text{otherwise,} \end{cases}$$

and $e_P$ is the order of the isotropy group of $P$.

## Proposition III.2.9-10.

1. $M_0(\mathrm{PSL}_2(\mathbb{Z})) = \mathbb{C}$
2. $M_k(\mathrm{PSL}_2(\mathbb{Z})) = \{0\}$ if $k = 2$ or $k < 0$.

## Corollary (Dimension Formulas)

*For positive even integers k We have*

$$\dim M_k(\mathrm{PSL}_2(\mathbb{Z})) = \begin{cases} \lfloor k/12 \rfloor, & \text{if } k \equiv 2 \mod 12, \\ \lfloor k/12 \rfloor + 1, & \text{if } k \not\equiv 2 \mod 12. \end{cases}$$

*and*

$$\dim S_k(\mathrm{PSL}_2(\mathbb{Z})) = \begin{cases} 0, & \text{if } k = 2, \\ \lfloor k/12 \rfloor - 1, & \text{if } k \equiv 2 \mod 12,\ k \geq 14, \\ \lfloor k/12 \rfloor, & \text{if } k \not\equiv 2 \mod 12. \end{cases}$$

## Proposition III.2.9-12.

1. $S_k(\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{C}\Delta$
2. $M_k(\mathrm{PSL}_2(\mathbb{Z})) = S_k(\mathrm{PSL}_2(\mathbb{Z})) \oplus \mathbb{C}G_k$, for $k \geq 4$.
3. For any $f \in M_k(\mathrm{PSL}_2(\mathbb{Z}))$,

$$f = \sum_{4i+6j=k} c_{i,j} E_4^i E_6^j, \quad \text{for some } c_{i,j} \in \mathbb{C}.$$

4. The elliptic $j$-funEisenstein seriesction gives a bijection from $X_0(1)$ to $\mathbb{P}^1(\mathbb{C})$.
5. The field of meromorphic functions on $X_0(1)$ is $\mathbb{C}(j)$.