Diagonalization — Spectral Theorem

In earlier discussion, we have seen the diagonalizable operators. For these operators, it is necessary and sufficient for the vector space to possess a basis of eigenvectors. In this note, we will consider the diagonalibility of a linear operator of a finite-dimensional inner product space over K ($K = \mathbb{R}$ or $\mathbb{C}$). In the case of inner product spaces, we are seeking conditions that guarantee that the inner product space has an orthonormal basis of eigenvectors.

In the following, we will first discuss the case of square matrices and then state general results for linear operators.

Throughout, we denote $K = \mathbb{R}$ **or** $\mathbb{C}$ and denote A to be an n-by-n real matrix or a complex matrix. Define

$$^*A := {}^t\overline{A}$$

to be the **conjugate transpose** or **adjoint** of the matrix $A \in M_n(\mathbb{C})$. (If $A \in M_n(\mathbb{R})$, $^*A := {}^tA$.)

1. Schur's Theorem

Lemma 1. *If A has an eigenvalue, then so does $^*A$.*

Theorem 1 (Schur)**.** *Suppose the characteristic polynomial of A splits over K. Then there exists an orthonormal basis β for K^n such that A can be represented by a upper trianglular matrix with respect to β.*

Corollary 1. *Suppose the characteristic polynomial of A splits over K.*

(1) *If $K = \mathbb{C}$, then A is unitarily equivalent to a complex upper triangular matrix.*
(2) *If $K = \mathbb{R}$, then A is orthogonally equivalent to a real upper triangular matrix.*

2. Normal and Self-adjoint matrices

Definition 1. *We say that A is* **normal** *if $^*AA = A^*A$, A is* **self-adjoint** *or* **Hermitian** *if $A = {}^*A$.*

Theorem 2. *Let A be a normal matrix. Then the following statements are true.*

(1) $\langle A\vec{v}, A\vec{v}\rangle = \langle {}^*A\vec{v}, {}^*A\vec{v}\rangle$, *for all $\vec{v} \in K^n$.*
(2) *$A - cI_n$ is normal for any $c \in K$.*
(3) *If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of $^*A$. Precisely, if $A\vec{v} = \lambda\vec{v}$, then $^*A\vec{v} = \overline{\lambda}\vec{v}$.*
(4) *If λ_1 and λ_2 are distinct eigenvalues of A with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$, then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.*

Theorem 3. *Let $A \in M_n(\mathbb{C})$. Then A is normal if and only if there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A.*

From the theorem, we see that, in $M_n(\mathbb{C})$, unitary matrices and Hermitian matrices are diagonalizable.

Lemma 2. *Let A be a self-adjoint matrix. Then*

(1) *Every eigenvalue of A is real.*
(2) *If $A \in M_n(\mathbb{R})$, i.e. A is symmetric, then the characteristic polynomial of A splits.*

Theorem 4. *Let $A \in M_n(\mathbb{R})$. Then A is self-adjoint (symmetric) if and only if there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.*

Corollary 2. *Let A be self-adjoint. If $\langle \vec{v}, A\vec{v} \rangle = 0$ for all $\vec{v} \in K^n$, then A is the zero matrix.*

3. The Spectral Theorem

In the previous section, we see that if a complex matrix A is unitarily equivalent to a diagonal matrix if and only if A is normal, and a real matrix A is orthogonally equivalent to a diagonal matrix if and only if A is symmetric. In this section, we will decompose a diagonalizable matrix interms of eignevalues of A and certain orthogonal projections.

Definition 2. *Let W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. A function $\phi : V \longrightarrow V$ is called the* **projection on** W_1 **along** W_2 *if, for $v = v_1 + v_2$ with $v_1 \in W_1$ and $v_2 \in W_2$, we have $\phi(v) = v_1$.*

In fact, one can show that $\phi^2 = \phi$ if and only if ϕ is a projection.

Proposition 1. *Let W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. Assume that $\phi : V \longrightarrow V$ is the projection on W_1 along W_2. Then*

(1) *ϕ is linear and $W_1 = \{v \in V : \phi(v) = v\}$.*
(2) *$W_1 = im\, \phi$ and $W_2 = \ker \phi$.*

Thus, we have $V = im\, \phi \oplus \ker \phi$.

Definition 3. *Let V be an inner product space, and $\phi : V \longrightarrow V$ be a projection. We say that ϕ is an* **orthogonal projection** *if $(im\, \phi)^\perp = \ker \phi$ and $(\ker \phi)^\perp = im\, \phi$.*

Theorem 5. *Let $A \in M_n(K)$. Then A is an orthogonal projection if and only if $A^2 = A = {}^*A$.*

Theorem 6 (The Spectral Theorem)**.** *Let $A \in M_n(K)$. Assume that A is normal if $K = \mathbb{C}$ and that A is symmetric if $K = \mathbb{R}$. Suppose that λ_1, λ_2, ..., λ_m are distinct eigenvalues of A. For each i, let E_i be the eigenspace of A corresponding to the eighenvalue λ_i, and let T_i be the orthogonal projection of K^n on E_i. Then the following statements are true.*

(1) $K^n = E_1 \oplus E_2 \oplus \ldots E_m$.
(2) *If W_i denotes the direct sum of the subspaces E_j for $j \neq i$, then $E_i^\perp = W_i$.*
(3) $T_iT_j = \begin{cases} T_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$
(4) $I_n = T_1 + T_2 + \cdots + T_m$.
(5) $A = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_m T_m$